MATHEMATICAL PROBLEM SOLVING WITH TECHNOLOGY BEYOND THE CLASSROOM: THE USE OF UNCONVENTIONAL TOOLS AND METHODS

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This paper addresses mathematical problem solving with technologies in a beyond school web-based competition. We aim to disclose the ways mathematical and technological knowledge are used and combined for solving the given problems. A specific conceptual framework for accounting both these components was developed. By means of the Mathematical Problem Solving with Technology model (MPST) we report the case of Marco, aged 13, solving and expressing a geometrical problem. His ability in perceiving affordances in the tools that he chose is in line with the efficient use he made of them in the development of mathematical understanding that was crucial for finding and expressing the solution. Results suggest that digital thinking and experience have to be seen as relevant as the mathematical cognitive resources.

INTRODUCTION

Mathematical Problem Solving has acquired the status of a research field within Mathematics Education over the last decades of the 20th century, after an intense research activity following the influential work of George Polya later developed by the seminal work of Alan Schoenfeld. The turn of the century brought new research objects and impetus, which have diverted the interest of the research community regarding this topic and, particularly, regarding the problem solving activity that occurs beyond the classroom (English & Sriraman, 2010). Recently, the constant availability and usage of sophisticated digital tools in out-of-school and beyond-school contexts are requiring new thinking about the sort of skills that may become especially important in the technological, global and interconnected society of the 21st century (Hoyles, Noss, Kent & Bakker, 2010). Thus, problem solving with new methods and new tools holds up as a central competence to meet the challenges of active life, as technological tools are altering ways of thinking and acting (Lesh, 2000).

This paper reports on a study that addresses students' mathematical problem solving with technology in a beyond-school context comprising of a web-based mathematics competition, named SUB14®. The competition is aimed at middle graders (12-14 years-old) of every school of the south of Portugal. Its Qualifying stage consists of solving a non-routine problem proposed every two weeks, either through e-mail or an online text editor available on the competition website. Participants may solve the problems using their preferred methods and tools but are explicitly required to report on their solving process and must offer a complete explanation of their reasoning.

Our main research goal is to uncover the generally unknown ways in which students

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use and combine their mathematical and technological knowledge outside the classroom, particularly when they are allowed to pick any digital tool of their choice and use it to achieve mathematical purposes. In this paper we hope to provide evidence for the claim that problem solving with digital tools can only be partially described by previous frameworks that took mathematical thinking and experience as the primary cognitive resources. What one must realize is that digital thinking and experience have to be seen as equally central and fundamental cognitive resources. Here we draw on the case of a student solving a geometrical problem, which shows much of his mathematical thinking going on when he handles similarity and ratios among circles inscribed in triangles, but also reveals specific actions and processes related to his use of digital tools for the analysis of the geometric figure presented in the problem. Thus, we seek to address this need to redesign and expand well-known earlier theoretical models and suggest more efficient ways to describe the connection between mathematical knowledge and the affordances of digital tools that solvers bring into the problem space.

MATHEMATICAL PROBLEM SOLVING WITH TECHNOLOGY

The prevailing theoretical models of the problem solving activity appear inadequate as tools for interpreting the role of technology and to explain the interaction between individuals' technological and mathematical competences in their problem solving activity (Santos-Trigo & Camacho-Machín, 2013). This has the development of a new specific conceptual framework that might account for both components of the problem solving process.

Solving a non-routine mathematical problem is here understood as the development of a productive way of thinking about a challenging situation (Lesh & Zawojewski, 2007) where the solver must adopt a mathematical point of view in order to carry out mathematization processes. Problem solving is also conceived as a synchronous process of mathematization and expression of mathematical thinking (Carreira, Jones, Amado, Jacinto & Nobre, in press). This means that solving a problem encapsulates both the required answer and the creation of an explanation for that answer.

In terms of student's interaction with digital media in performing complex tasks, such as non-routine problems, we draw on the concept of perception of affordances in the tools (Gibson, 1977). "Perceiving affordances is placing features, seeing that the situation allows a certain activity" (Chemero, 2003, p. 187). This suggests that the solver's effective use of a tool is grounded on the recognition of its particular features that will be useful for developing an approach to the problem. Affordances emerge from the relationship between the capabilities of the solver and the properties of the tool (Norman, 2013), insofar as one is not "specifiable in the absence of specifying the other" (Greeno, 1994, p. 338), which leads us to consider the impossibility of separating the solver's mathematical and technological competences.

The DigEuLit Project proposed a model that sets a list of processes performed while solving a task or problem that requires the use of a digital resource, comprising:

statement – clearly state the problem and the actions likely to be required; *identification* – identify the digital resources required to achieve the solution; *accession* – locate and obtain those digital resources; *evaluation* – assess the accuracy and reliability and relevance of the digital resources; *interpretation* – understand the meaning they convey; *organization* – organize them in ways that may enable the solution; *integration* – bring these resources together in relevant combinations; *analysis* – examine them using concepts and models that will enable the solution; *synthesis* – recombine them in new ways to achieve the solution; *creation* – create new knowledge objects, units of information or digital outputs that contribute to achieve the solution; *communication* – interact with others while solving the problem; *dissemination* – present the solution to others; *reflection* – consider the success of the task performed (Martin & Grudziecki, 2006, p. 257).

Although several actions in this list resemble well known problem solving models, a mathematical lens is needed. Alan Schoenfeld's (1985) model for describing students' mathematical problem solving performance seemed useful to this task. His five stage model comprises: *read* – time spent "ingesting the problems conditions"; *analysis* – attempt to fully understand the problem "sticking rather closely to the conditions or goals" that may include a selection of ways of approaching the solution; *exploration* – a "search for relevant information" that moves away from the context of the problem; *planning and implementation* – defining a sequence of actions and carrying them out orderly; *verification* – the solver reviews and assesses the solution (pp. 297-298).

Mathematical problem solving with technology (MPST)		
Grasp	Appropriation of the situation and the conditions in the problem, and early ideas on what it involves.	Communicate – Interact with relevant others whilst dealing with the problem or task.
Notice	Initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the digital tools that may be necessary.	
Interpret	Placing affordances in the technological resources in pondering mathematical ways of approaching the solution.	
Integrate	Combining technological and mathematical resources within an exploratory approach.	
Explore	Using technological and mathematical resources to explore conceptual models that may enable the solution.	
Plan	Outlining an approach to achieve the solution based on the analysis of the conjectures explored.	
Create	Carrying out the outlined approach, recombining resources in new ways to create new objects that convey both mathematical and technological understanding of the situation, which will contribute to solve-and-express the problem.	
Verify	Engaging in activities to explain or justify the solution achieved based on the mathematical and technological resources.	
Disseminate	Present the solutions or outputs to relevant others and consider the success of the problem-solving process.	

Table 1: Processes underlying mathematical problem solving with technology

By comparing and relating the processes proposed by Martin and Grudziecki and the stages identified by Schoenfeld, we reached a proposal of merging these two frameworks by means of fusing some of the processes of digital tool usage and segmenting some of the stages of mathematical problem solving (Jacinto & Carreira, to appear). Table 1 presents a summary of the processes involved in solving a mathematical problem with technology.

RESEARCH METHOD

The larger study from which we extract the data covered here focuses on students' use of freely chosen technological tools for solving and expressing the mathematical problems posed by SUB14. The explorative nature of the study demanded an interpretative approach that involved qualitative techniques for data collection and analysis (Quivy & Campenhoudt, 2008).

Data collection is based on two different sources: the solutions submitted by the participants throughout the Qualifying phase, and two clinical interviews that took place at the participants' home with the permission of their parents. The second interview, video-recorded, included the observation of the student while solving a problem posted at the competition's website and thinking aloud.

This paper reports on the case of Marco (pseudonym) who usually resorts to a variety of technological tools to solve the problems and present his explanations. The data refer to the observation of Marco while working on a problem. We used NVivo for organizing the data, transcribing the interviews, segmenting and coding. Marco was asked to choose one out of three problems posted for this purpose on the SUB14 website, and to solve it by performing as closely as possible to his usual problem solving process in the competition. Marco chose to solve the problem "Decorative Drawing" (Figure 1) and resorted to several technological tools during the process. Marco's processes of problem solving-and-expressing with technology will be considered and interpreted through the lens of the MPST analytical framework. The following section presents a summary of our findings.

The picture shows a decorative drawing that will be used in the construction of a stained glass window. The equilateral triangle has a height of 12 cm. The circles are all tangent to the triangle and also each small circle is tangent to the large circle. Which is the radius of the smaller circle?

Don't forget to explain your problem solving process!

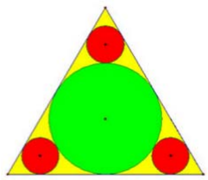


Figure 1: Statement of the problem 'Decorative Drawing' chosen by Marco

DATA ANALYSIS

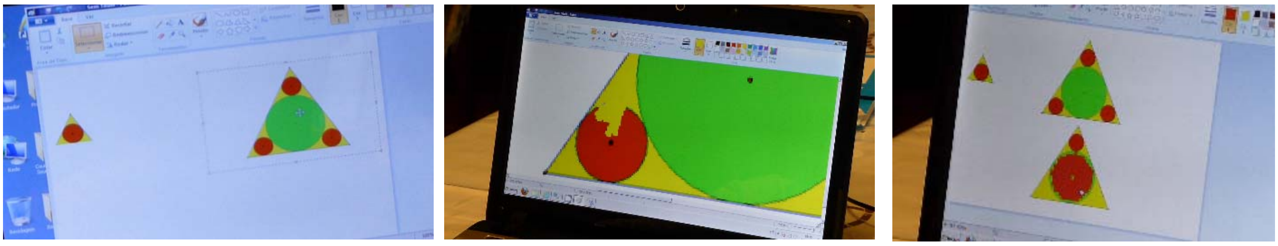
Marco is a 13 years-old student, very enthusiastic about digital tools. At school, his math teacher usually uses the whiteboard and sometimes takes the class to a computers room with specific tasks to perform. Marco is quite familiar with a diversity of digital tools; at school, in particular, he learned to use GeoGebra, while studying geometric transformations. Usually, he submits his answers to SUB14 in a spreadsheet file.

Marco solving the problem "A Decorative Drawing"

After carefully analysing the three problems on the SUB14 website, Marco chose to solve his favourite. When asked about his reasons, he explains: "It has to do with triangles and stuff like that, besides in the 7th grade I got 100 [%] in both tests (…) I studied congruent triangles and such…" His choice seems grounded on the immediate recognition of the mathematical notions that may be necessary to solve the problem and, simultaneously, on how familiar and self-confident he feels (*grasp*).

As Marco starts to interact with the figure shown in the problem and displayed at the competition's website; he develops several arguments that lead to a conjecture about the solution. Initially, this includes attempts to understand what the problem involves (*notice*), and in each argument he makes considerations about mathematical ways of approaching the solution (*interpret*). The sequence of arguments was as follows: i) "Since the triangle is equilateral, if I reach the circle in the centre, I might get to the others"; ii) "It's like it divides in half. Dividing from each vertex to the midpoint of the opposite side; iii) "It has 12 cm. At the middle of the triangle it is not 12, for sure. But it could be 4. Dividing these parts… Because they are tangent... I can tell they are the same length". While Marco is thinking aloud, he 'interacts' with the figure on the screen: he points, estimates distances, hides areas with his hands. Developing a visual approach to the problem, Marco considers the possibility of decomposing the figure mentally simulating different transformations – cutting, reorganizing, and recolouring.

He finally conjectures: "If we draw a triangle here (…) this is an enlargement of the other triangle. If it is 12, 12 divided by 3, [is] 4… Maybe the radius of the smaller circle is 2". By this, Marco is considering the construction of a small equilateral triangle at the top of the given one. This triangle is obtained using the Snipping Tool: Marco sets up a region at the top of the original triangle and saves the new image as a separate file. He uses the same process to obtain an image similar to the original triangle (*integrate*).



A) Pastes the two images cropped. B) Covers the red circles with yellow. C) Paints the central circle in red.

Figure 2. Three steps of the image processing with MS Paint

He then pastes both these triangles in MS Paint (Figure 1A) and attempts to overlap the two of them. As the images have a white background, the overlaping is not satisfactory for him, so he decides to edit the original triangle to make it similar to the smaller one by removing the red circles (Figure 2B) and recoloring the central circle, initially green, as red (Figure 2C). Marco is developing and exploring a conceptual model for explaining the similarity of these two triangles (*explore*) that will guide him in the construction of the solution. At a certain point, he opens a blank spreadsheet. Then, never 'leaving' the computer screen and without resorting to any other exterior tool – neither a notepad nor a pencil – Marco keeps moving between the website, which displays the problem, the image processing tools, and the spreadsheet, where the solution will be expressed. (*plan*). The original image and the two manipulated figures (Figure 4C) become a mathematical argument that he resorts to while assembling his answer in the spreadsheet. These images support his understanding of the problem specifically the way he envisions the similarity between the two triangles. By integrating mathematical ideas, related to similarity of triangles, and the deconstruction of the triangle by means of the editing tools, he reaches a conceptual model of the situation (*create*). Actually, the spreadsheet contains the three images and a verbal text where he reports the whole process.

> …that smaller triangle is a reduction of the larger triangle; since it is a reduction all I have to do is 12:3 (which is twice the radius of the green circle plus the height of the smaller triangle) and I got 4, which is the radius of the green circle; as the smaller triangle is a reduction of the larger one and its height is 4, to obtain the radius of the red circle one must divide 4:3 which is 4/3. (Excerpt of Marco's written part of the solution).

As he engages in writing an explanation and analysing the images processed (*verify*) Marco reaches the solution to the problem – the radius of the smaller circle is 4/3 – which is actually different from the conjecture that he formulated at the beginning and that guided his approach. Throughout the process, Marco occasionally interacts with the researcher for clarification of wording (*communicate*) and, when finished, he submitted his answer to the competition using the editor embedded in the SUB14's webpage (*disseminate*).

Marco's initial activity seems to have a recurrent nature, where each argument is formulated as he tries to make sense of the mathematics that may be relevant (notices) and considers mathematical ways of approaching the solution (interprets) while he interacts with the figure on the screen. This cyclic activity leads Marco to a final conjecture – "the radius of the smaller circle is 2" – which is his first answer to the problem and will trigger the subsequent exploration activity. Marco's success in achieving the solution to the problem seems to be related to his ability in recognizing the affordances of the selected tools, which empower his thinking process, and, ultimately, influence the expression of his reasoning. Starting with exploring the first conjecture, Marco's elaboration of images in the graphic environment leads him to find the correct ratio of similarity. The move to the spreadsheet environment supports the combination of objects because it affords an easy organization of images and textual inscriptions (the images move freely, formatting is easy as well as cell merging).

DISCUSSION AND CONCLUDING REMARKS

The analysis presented above shows that unconventional tools, such as Paint or the Snipping Tool, can be used efficiently to develop mathematical understanding that becomes crucial for finding and expressing a solution to a problem. Cropping, reconstructing or recolouring images lead to the creation of new objects that convey mathematical and technological understanding of the situation. These new objects not only contribute decisively to finding the answer, but they also become a crucial part of the solution as they allow to establish a roadmap to the approach developed (Lesh &

Doerr, 2003).

Moreover, the effectiveness of these technological tools as 'problem solving tools' seems mainly arising from the digital representations they afford, which allow manipulating images and for this reason can foster a geometrical interpretation of the situation that, in turn, enhances the development of a conceptual model. Marco's elaboration of images played a paramount role in every phase of the processes of mathematization and expression of thinking. In fact, the development of his mathematical thinking seems to take advantage of the affordances of the tools that he found helpful in finding the solution to the problem. This is in line with the theory of affordances, namely the most recent developments that contribute to explain human-computer interaction (Norman, 2013).

Additionally, the youngster's constructions and explanations are crucial elements that assume a double role: they simultaneously support the finding of the solution and the reporting of the answer. Thus, this case highlights the artificiality of the boundaries between solving the problem (i.e., the processes followed in obtaining the solution) and constructing the answer (i.e., the solution in the file to be submitted), since the mathematical thinking is developed in a continuum and is refined whilst the explanation is being produced. This strengthens the idea that solving and expressing are simultaneous mathematizing activities. Hence, solving-and-expressing is a way of accounting for the youngsters' mathematization processes, particularly when technological and mathematical knowledge come into play in the development of an approach to the problems (Carreira et al., in press).

While there are powerful models that account either for the processes of mathematical problem solving, or for the processes taking place with digital tools in general tasks, the MPST model provides the means for describing problem solving with technology, letting the combination between mathematical and technological knowledge and skills to emerge throughout the whole process. The levels of description achieved within this model, grounded on the more general conceptual framework, allow to acknowledge the role of technological tools in mathematical problem solving, even when such tools appear deprived of mathematical affordances. Today's real world problem solving activity, highly impregnated with digital tools, requires such a framework with a broader scope, capable of supporting the specificities of the digital tools considering their affordances in terms of the mathematical thinking needed for achieving an elegant solution and communicating it effectively.

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